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# On the Theory and Application of the Dielectric Post Resonator

MARIAN W. POSPIESZALSKI

***Abstract*—This short paper deals with the modes of the dielectric post resonator when $\varepsilon_r$ is large. The normalized frequency $F_0 = (\pi D/\lambda_0)\sqrt{\varepsilon_r}$ as a function of $D/L$ is discussed. The simple approximate expressions for the resonant frequencies of the lower order modes are given. The properties of the $TE_{011}$ mode are discussed in detail from the point of view of its application to the measurement of the complex permittivity of microwave dielectrics. Curves and expressions for fast and simple determination of the maximum measurement errors are given.**

## I. Introduction

In this short paper, the resonant properties of the structure shown schematically in Fig. 1 will be discussed. The cylindrical sample of a low-loss high-$\varepsilon_r$ dielectric material is placed between two parallel conducting plates. This structure, known in the literature as the dielectric post resonator, was applied by Hakki and Coleman [1], and later by Courtney [2], to measurements of the complex permittivity and complex permeability of microwave insulators. It was also used to provide high RF field concentrations on ferrite crystals [3], [5]. Present availability of the low-loss high-$\varepsilon_r$ temperature compensated ceramics (for instance, [6], [7]) should permit introduction of this structure as an element of microwave filters, oscillators, etc. Information on the modes of the dielectric post resonator that can be found in the literature are valid only for certain values of $\varepsilon_r$ [2]–[4]. A mode chart, together with approximate expressions for the resonant frequencies of the lower order modes, valid for all cases when $\varepsilon_r \geq 10$ is given. Though the properties of the particular $TE_{011}$ mode were discussed in detail [1], [2], some important simplifications can be carried out, as is shown in the following.

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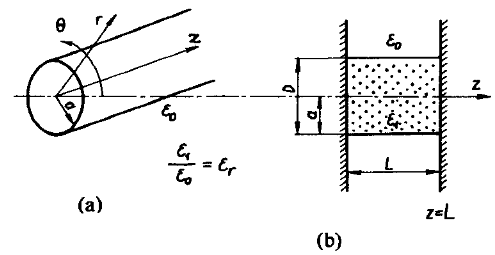


Fig. 1. Dielectric rod placed between two infinite parallel perfectly conducting plates.

## II. Dielectric Post Resonator

### *Basic Relations*

Referring to Fig. 1, the dielectric post resonator can be described as a dielectric rod transmission line short circuited at both ends. The characteristic equation for the normal modes of the structure [8], [9] can be written in the form

$$(J^+ + K^+)\left(J^- - \frac{K^-}{\varepsilon_r}\right) + (J^- - K^-)\left(J^+ + \frac{K^+}{\varepsilon_r}\right) = 0 \tag{1}$$

$$\left(\frac{\pi D}{\lambda_0}\right)^2 \varepsilon_r = u^2 + \left(\frac{\pi D l}{2L}\right)^2 \tag{2}$$

$$w^2 = \left(\frac{\pi D l}{2L}\right)^2 - \left(\frac{\pi D}{\lambda_0}\right)^2 \tag{3}$$

and

$$J^- = \frac{J_{n-1}(u)}{uJ_n(u)} \qquad J^+ = \frac{J_{n+1}(u)}{uJ_n(u)}$$

$$K^- = \frac{K_{n-1}(w)}{wK_n(w)} \qquad K^+ = \frac{K_{n+1}(w)}{wK_n(w)}$$

where $J_n$ is the Bessel function of the first kind of $n$th order, $K_n$ is the modified Hankel function of $n$th order, $l$ is the number of half-wavelengths in the cavity along the axial direction, $D$ and $L$ are the diameter and length, respectively, $\varepsilon_r$ is the relative dielectric constant, and $\lambda_0$ is the free-space wavelength corresponding to the resonant frequency $f_0$. The solution of the preceding set of equations can be most conveniently presented in the form $F_0^2 = (\pi D/\lambda_0)^2\varepsilon_r = f_1(D/L)$. $F_0$, which we call a normalized frequency variable, is very small dependent on $\varepsilon_r$ when $\varepsilon_r$ is much greater than unity. Indeed, for all values $D/L > 0$ and $\varepsilon_r \to \infty$, the asymptotic form of the previous equations is

$$(J^+ + K^+)J^- + (J^- - K^-)J^+ = 0 \tag{4}$$

$$F_0^2 = u^2 + w^2 \tag{5}$$

$$w = \frac{\pi D l}{2L}. \tag{6}$$

This is very well illustrated by the mode chart for the lower order modes shown in Fig. 2. The solutions for $\varepsilon_r \geq 500$ cannot be distinguished graphically. The largest difference in the range of interest for these cases is 0.46 percent for the $TE_{011}$ mode and $(D/L)^2 = 0.5$.

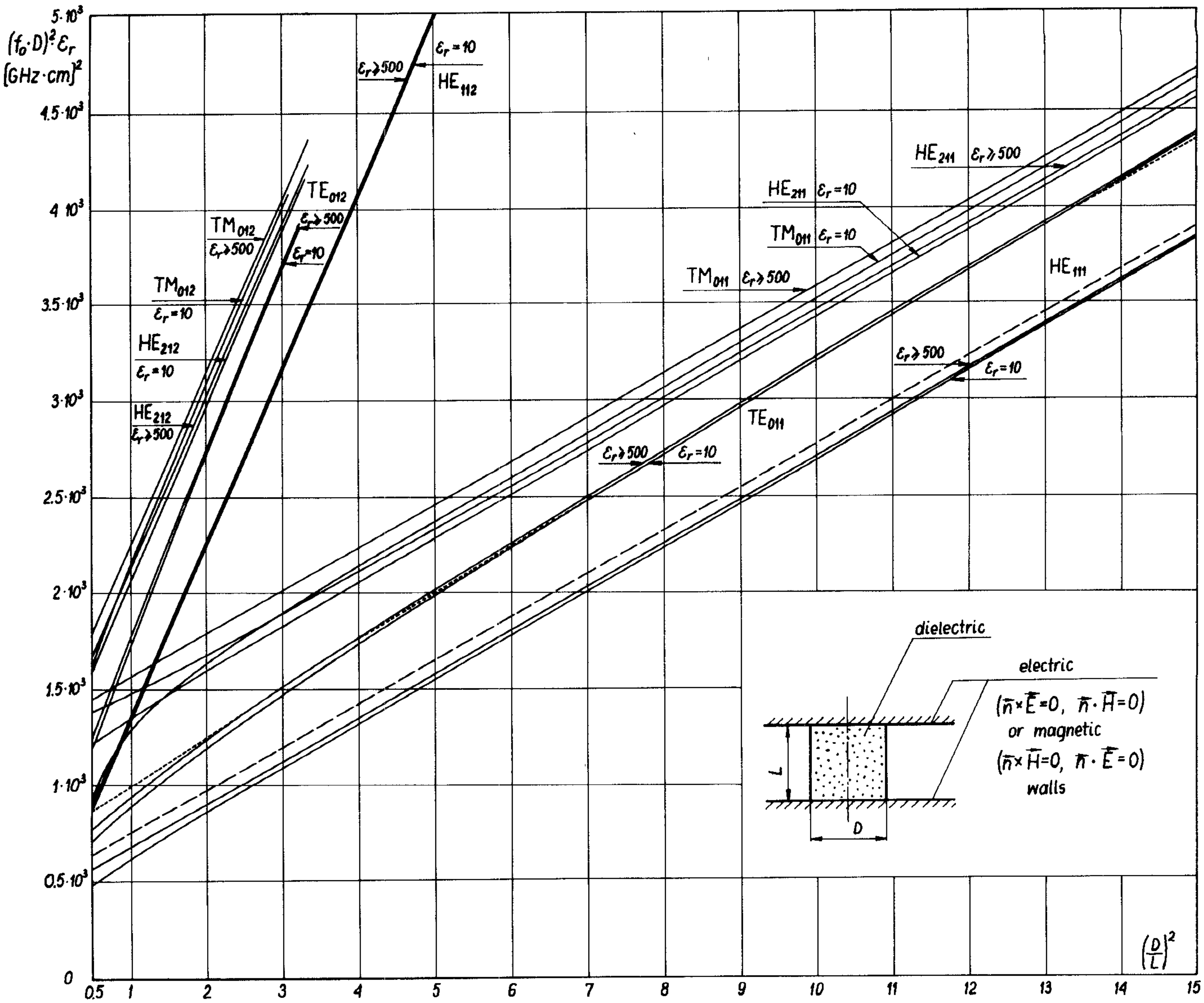


Fig. 2. Mode chart for the dielectric post resonator.

*Approximate Expressions*

For the $HE_{11l}$, $HE_{21l}$ modes the solutions of (1)–(3) in the very broad range of $\varepsilon_r$ and $D/L$ are almost straight lines. Thus very simple approximate formulas can be derived. The plots for $HE_{11l}$ modes, if $\varepsilon_r \geq 500$ and $1 \leq (D/L)^2 \leq 15$, can be approximated by

$$F_0^2 = (2.21)^2 + \left(\frac{\pi D l}{2L}\right)^2. \tag{7}$$

The accuracy is better than 0.7 percent. Similarly, for $HE_{211}$ we get

$$F_0^2 = (3.65)^2 + \left(\frac{\pi D l}{2L}\right)^2. \tag{8}$$

The accuracy of the expression (8) is better than 0.7 percent if $\varepsilon_r \geq 500$ and $2 \leq (D/L)^2 \leq 15$.

For circularly symmetric modes, the characteristic equation (1) has very simple forms.

1) For $TM_{0ml}$,

$$\frac{J_1(u)}{uJ_0(u)} = -\frac{1}{\varepsilon_r}\frac{K_1(w)}{wK_0(w)}. \tag{9}$$

2) For $TE_{0ml}$,

$$\frac{J_1(u)}{uJ_0(u)} = -\frac{K_1(w)}{wK_0(w)}. \tag{10}$$

If $\varepsilon_r$ is large enough, the solutions of (9) for $w > 0$ can be approximated by the solutions of

$$J_1(u) = 0. \tag{11}$$

Thus an approximate expression for the resonant frequencies of $TM_{0ml}$ modes has the following form:

$$F_0^2 = \rho_{1m}^2 + \left(\frac{\pi D l}{2L}\right)^2 \tag{12}$$

where $\rho_{1m}$ is the $m$th greater than zero solution of (11). For instance, for the $TM_{011}$ mode the accuracy of the expression (12) is better than 2 percent if $\varepsilon_r \geq 20$ and $(D/L)^2 \geq 1$, and better than 0.5 percent if $\varepsilon_r \geq 100$ and $(D/L)^2 \geq 1$. For higher order TM modes, the accuracy is even better. It should be noted that expression (12) is also an asymptotic solution for $HE_{2ml}$, $TE_{0ml}$, $TM_{0ml}$ modes, when $D/L$ goes to infinity.

The solutions for $TE_{0ml}$ modes can be approximated by

$$F_0^2 = \rho_{1m}^2 \left(1 - \frac{1}{\sqrt{\rho_{1m}^2 + \left(\frac{\pi D l}{2L}\right)^2}}\right)^2 + \left(\frac{\pi D l}{2L}\right)^2. \quad (13)$$

This expression is illustrated in Fig. 2 by the dotted line. The accuracy is better than 0.5 percent if $\varepsilon_r \geq 500$ and $3 \leq (D/L)^2 \leq 15$. The derivation of the expression (13) is given in [10].

In many papers (for instance, [11]–[13]) the dielectric resonator was treated as a length of uniform waveguide with magnetic walls (i.e., $\boldsymbol{n} \times \boldsymbol{H} = 0$, $\boldsymbol{n} \cdot \boldsymbol{E} = 0$). It is obvious that under this assumption only TE and TM modes exist. In the case of the dielectric post resonator, the expressions for $TM_{0ml}$ modes take exactly the same form as (12). On the other hand, for the $TE_{0ml}$ modes we get

$$F_0^2 = \rho_{0m}^2 + \left(\frac{\pi D l}{2L}\right)^2 \quad (14)$$

where $\rho_{0m}$ are $m$th roots of $J_0$.

Equation (14) is represented graphically in Fig. 2 by the broken line. It represents an asymptotic solution for $HE_{1ml}$ modes when $D/L$ goes to infinity. Thus the magnetic wall assumption in the case of cylindrical structures can lead to great errors for TE modes, giving at the same time good results for TM modes. It can serve as an illustration of the discussion presented by Van Bladel [14]. Using his terminology, we should refer to the TM modes of the dielectric post resonator to be of the "confined" type. Consequently, all the other modes, e.g., $TE_{0ml}$, $HE_{nml}$, $EH_{nml}$, are "nonconfined" modes.

## III. The Dielectric Resonator Method of Measuring the Permittivity of Microwave Insulators

In this method [1], [2] the dielectric sample forms the microwave resonator. Thus measuring sample dimensions $D$ and $L$, resonant frequencies $f_0$, and unloaded $Q$ factor $Q_u$ of the cavity operating in the known mode and with the known surface resistivity $R_s$ of the metallic plates, one can determine the relative dielectric constant $\varepsilon_r$ and loss tangent $\tan \delta$ of the dielectric. The $TE_{011}$ mode is always chosen because the only existing transverse component of the electric field $E_\theta$ (Fig. 1) vanishes at the dielectric metal interfaces. Thus an error due to the air-gap effect is practically eliminated. This method is in many aspects described in [1], [2]. We only point out some simplifications which result from the aforementioned theory.

### *Identification of the $TE_{011}$ Resonance*

It is seen from the mode chart presented in Fig. 2 that the $TE_{011}$ mode can be identified as a second low-frequency mode. It is also seen that mode frequency separation is poor when the value of the $D/L$ ratio is too small or too large. Thus it is suggested to use a sample having a $D/L$ ratio ranging between 1 and 3. It should be noted that the resonant frequency of the $TE_{011}$ mode is relatively insensitive to small changes of the air-gap dimensions in comparison with other modes, i.e., $HE_{111}$, $HE_{211}$, $TM_{011}$, $HE_{112}$.

### *Accuracy of $\varepsilon_r$ Measurement*

Assuming that the normalized frequency $F_0$ is only a function of $D/L$ for $\varepsilon_r \gg 1$, say $F_0^2 = f_1(D/L)$, the equation for the maximum relative measurement errors $|\Delta\varepsilon_r/\varepsilon_r|$ gets very simple:

$$\left|\frac{\Delta\varepsilon_r}{\varepsilon_r}\right| = 2 \cdot \left|\frac{\Delta f_0}{f_0}\right| + f_2\left(\frac{D}{L}\right) \cdot \left|\frac{\Delta L}{L}\right| + \left[2 - f_2\left(\frac{D}{L}\right)\right] \cdot \left|\frac{\Delta D}{D}\right| \quad (15)$$

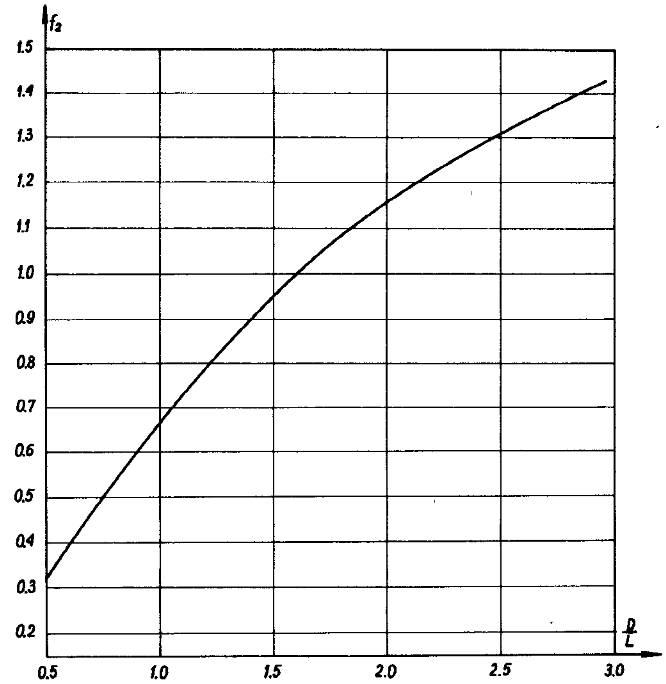


Fig. 3. Plot of the function $f_2$ [see expression (15)] given to determine the error in $\varepsilon_r$ measurements.

where

$$f_2\left(\frac{D}{L}\right) = \frac{f_1'\left(\frac{D}{L}\right) \cdot \frac{D}{L}}{f_1\left(\frac{D}{L}\right)}$$

and $f_1'(D/L)$ is the derivative of $f_1(D/L)$. $|\Delta f_0/f_0|$, $|\Delta L/L|$, $|\Delta D/D|$, are maximum relative measurement errors of the resonant frequency $f_0$, length $L$, and diameter $D$, respectively.

The plot of the function $f_2$ computed numerically from (4)–(6) is shown in Fig. 3. Equation (15) together with Fig. 3 allow a prediction of the accuracy of the $\varepsilon_r$ measurement which depends on the $D/L$ ratio, the sample accuracies, and the accuracy in frequency measurement.

### *Accuracy of the Tan $\delta$ Measurement*

Hakki and Coleman [1] give expression for $\tan \delta$ in terms of the sample dimensions $D$ and $L$, the relative dielectric constant $\varepsilon_r$, the unloaded $Q$ factor of the cavity $Q_u$, and the surface resistivity $R_s$ of metallic plates, if the resonator is working in the $TE_{011}$ mode. Their expression of rather complicated mathematical form can be simplified to

$$\tan \delta = \frac{1 + \frac{1}{\varepsilon_r} f_3}{Q_u} - \frac{R_s}{Z_f} f_4 \quad (16)$$

where $Z_f$ is the intrinsic impedance of the dielectric under test, i.e.,

$$Z_f = \sqrt{\frac{\mu_0}{\varepsilon_0 \varepsilon_r}}$$

and $f_3$ and $f_4$ are functions of $D/L$ and $\varepsilon_r$.

Fig. 4 gives plots of $f_3$ and $f_4$ in terms of $D/L$ for $\varepsilon_r = 10$ and for $\varepsilon_r \to \infty$. As these functions are not very dependent on $\varepsilon_r$, the presented graphs give good accuracy in most practical cases when $\varepsilon_r \geq 10$. It can be seen that the numerator of the first term in the expression (16) is almost equal to unity for $D/L > 1$ and

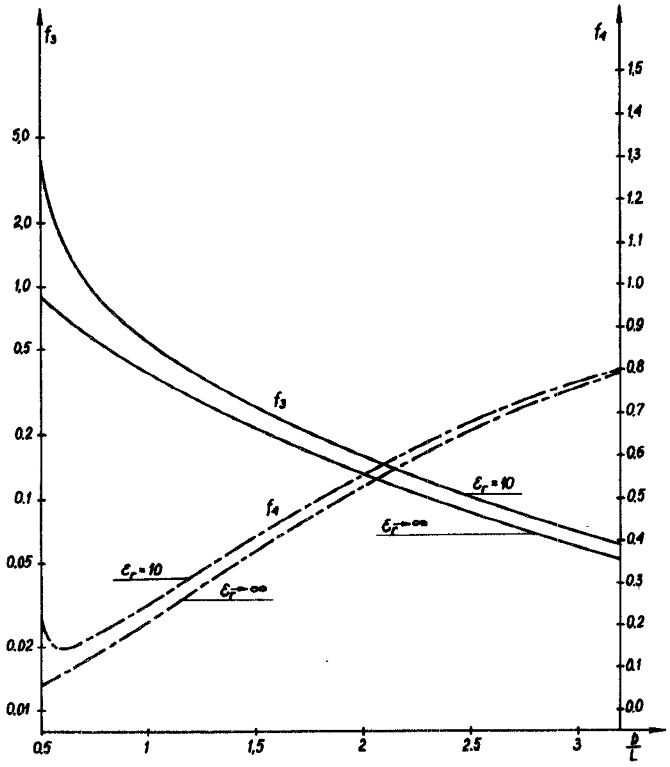


Fig. 4. Plots of the functions $f_3$ and $f_4$ [see expression (16)] given to determine the error in tan $\delta$ measurements.

$\varepsilon_r > 10$. Thus the influence of the accuracy of the $Q$-factor measurement on the accuracy of the tan $\delta$ measurement is evident. The second term in (16) represents the effect of the losses caused by the currents induced by RF fields in the metallic plates. It is well known that surface resistivity depends very much on surface finishing and may have a different value from that predicted theoretically. In this structure we cannot measure $R_s$ directly, unless we have a standard specimen, the loss tangent of which is accurately known. Equation (16) shows that the error of the tan $\delta$ measurement caused by poor knowledge of $R_s$ is small if $R_s/Z_f$ and $D/L$ ratios are small (see Fig. 4). Care must be taken when using samples with very small $D/L$ ratios due to the poor mode separation.

Many samples made of the same piece of dielectric having different $D/L$ ratios may have the same resonant frequency $f_0$. Thus measurements made on at least two samples allow us to find values of tan $\delta$ and $R_s$. This case should be taken into consideration if the accuracy of the tan $\delta$ measurement predicted previously is unsatisfactory.

A more detailed discussion of the dielectric post resonator properties from the point of view of its application to microwave measurements of dielectrics was given by the author in [15].

## IV. Conclusion

The dielectric post resonator, in view of the present tendency to miniaturize microwave circuits, can be a very attractive and easy to design structure. Cavities with $Q$ factors of 1000 ÷ 2000, a frequency stability better than $10^{-5}/°\text{C}$, and a dielectric volume of about 0.5 $\text{cm}^3$ can be easily built at $X$-band frequencies. Curves and equations given in this short paper simplify the use of the dielectric post resonator in the measurement of the permittivity in the microwave range.

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